INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: 2024-25

B. Stat. I Year

Vectors and Matrices II

Date: 25/02/2025 Maximum Marks: 40 Duration: 2 Hours

Note: Give proper justification to all your answers.
State clearly all the results you are using.

Notation: $\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of all positive integers.

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(1) Let F be a field and $n \in \mathbb{N}$. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of F^n. Consider a linear map $T : F^n \to F^n$. For $\alpha_1, \alpha_2, \ldots, \alpha_n \in F^n$, define

$$D_T(\alpha_1, \alpha_2, \ldots, \alpha_n) = \det(T\alpha_1, T\alpha_2, \ldots, T\alpha_n).$$

(a) Show that D_T is an alternating n-linear function.

(b) Let $c = \det(Te_1, Te_2, \ldots, Te_n)$. Show that for any n vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have

$$\det(T\alpha_1, T\alpha_2, \ldots, T\alpha_n) = c \det(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

(c) If $\mathcal{B}$ is any basis for F^n and A is the matrix of T in the basis $\mathcal{B}$, show that $\det A = c$. [10]

(2) Let M be an $n \times n$ integer matrix. That is, the entries of M are all integers. Show that the following are equivalent.

(a) For each $b \in \mathbb{Z}^n$, the system $Mx = b$ has a unique solution over $\mathbb{R}$ and the solution belongs to $\mathbb{Z}^n$.

(b) M^{-1} exists and is an integer matrix.

(c) $|\det M| = 1$. [8]

[Hint. Prove (a) $\Leftrightarrow$ (b) and (b) $\Leftrightarrow$ (c). For (a) $\Rightarrow$ (b), take $b = e_j$.]

(3) Consider the subspaces S and T of $\mathbb{R}^5$, where

$$S = \{(x_1, x_2, x_3, x_4, x_5) : x_1 = x_3 = x_5 \text{ and } x_2 + x_4 = 0\}$$

and

$$T = \{(x_1, x_2, x_3, x_4, x_5) : x_1 = x_2 = x_3 \text{ and } x_4 = 0\}.$$

Equip $\mathbb{R}^n$ with the standard inner product. Verify that $(S + T)^\perp = S^\perp \cap T^\perp$. [8]

(4) Let V be the vector space of all real polynomials of degree at most 5 with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx, \quad f, g \in V.$$

Let $W = \text{span}\{f_1, f_2, f_3\}$, where

$$f_1(x) = x, f_2(x) = x^3, f_3(x) = x^5.$$

Obtain an orthogonal basis of W using Gram-Schmidt orthogonalization process. [7]

(5) Let

$$A = \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}.$$

(a) Find all the eigenvalues of A.

(b) Find a basis for the eigenspace corresponding to each eigenvalue of A. [9]