Indian Statistical Institute, Kolkata

Final examination: Second semester 2024 -'25

B.Stat I year

Subject	:	**Analysis II**
Time	:	3 hours
Maximum score	:	50

Work independently. Attempt all the problems. Please use a new page to answer each problem and make sure that the problem number in the margin can be read, even after stapling. If you attempt the same problem several times, please strike out all the attempts except the correct one before submitting your answer script. Justify every step in order to get full credit of your answers. All arguments should be clearly mentioned on the answer script. Points will be deducted for missing or incomplete arguments.

(1) Let f be a Riemann integrable function over the interval $[a, b]$, not necessarily continuous. Prove that there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ of continuous functions on $[a, b]$ such that $\lim_{n\to\infty} \int_a^b |g_n(x) - f(x)|dx = 0$.

[7 marks]

(2) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that for every $x \in [a, b]$,

$$\int_a^x \left(\int_a^u f(t)dt \right) du = \int_a^x (x - u)f(u)du$$

[4 marks]

(3) (i) Let $f_n(x) = nx(1 - x^2)^n$, $x \in [0, 1]$. Show that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges to a function f integrable on $[0, 1]$ but the convergence is not uniform on $[0, 1]$.

(ii) Prove that the series of functions $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n(n+x)}$ is uniformly convergent on $[0, 1]$. If $f(x)$ is the sum of the series, prove that f is continuous on $[0, 1]$.

[4 + 4 = 8 marks]

(4) For $n \in \mathbb{N}$, define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = xe^{-nx}$.

(i) Prove that $\sum_{n=1}^{\infty} f_n$ is convergent pointwise.

(ii) Show that $\left[\sup_{x\geq 0} \sum_{k=n+1}^{2n} f_k(x)\right] \geq e^{-2}$ for all $n \in \mathbb{N}$. (*Hints: Consider the value at $x = \frac{1}{n}$.*)

(iii) Prove that $\sum_{n=1}^{\infty} f_n$ is not uniformly convergent. (*Hint: Use part (ii).*)

[3 + 4 + 3 = 10 marks]

(5) (i) Does the integral $\int_0^1 \frac{\ln x}{1-x} dx$ converge or diverge? Justify your answer.

(ii) Show that $\int_0^\infty \frac{\sin x \cos x}{x} dx = \frac{\pi}{4}$.

[3 + 3 = 6 marks]

(6) Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$.

(a) Calculate the Fourier coefficients of f, and show that

$$\widehat{f}(n) = \begin{cases} \pi/2 & \text{if } n = 0, \\ \dfrac{-1+(-1)^n}{\pi n^2} & \text{if } n \neq 0. \end{cases}$$

(b) What is the Fourier series of f in term of sines and cosines?

(c) Taking $\theta = 0$, prove that

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[3 + 2 + 4 = 9 marks]

(7) Show that if p is any polynomial, then

$$\lim_{x \to 0} p(1/x) e^{-1/x^2} = 0$$

Hence show that $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is infinitely differentiable. Obtain the Taylor series of f around 0.

[3 + 3 = 6 marks]