Indian Statistical Institute
Back paper Examination: 2025-26

Course Name: M. Math 1st year
Subject Name : Analysis of several variables
Maximum Marks: 100, Duration: Three hours
Date: 18.12.2025

All questions carry 10 marks. Answer all questions

1. If $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is defined by $f(A) = A^{-1}$, prove that for all B in $M_n(\mathbb{R})$,
$$Df(A)(B) = -A^{-1}BA^{-1}.$$

2. Suppose U is an open set in $\mathbb{R}^n$ and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is a differentiable function. Let x_0 be a fixed element in U where at least one partial derivative of f is not equal to zero. Let us denote by $S^{n-1}$the set
$$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}.$$
Define a function
$$g : S^{n-1} :\to \mathbb{R},\ g(v) = |D_v f(x_0)|.$$
Prove that g attains its maxima at the points $\pm \frac{\nabla f(x_0)}{\|\nabla f(x_0)\|}$.

3. Prove that for all $A \in \mathrm{GL}_n(\mathbb{R})$ and for all $X \in M_n(\mathbb{R})$,
$$D(\det)(A)(X) = \det(A)\mathrm{Tr}(A^{-1}X).$$

4. Suppose U is an open convex set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a C^2-function. Prove that for all a in U and h in $\mathbb{R}^n$ such that $\|h\|$ is sufficiently small, there exists a real-valued function E defined on an open set containing zero such that
   (a) $\|E(h)\| \to 0$ as $\|h\| \to 0$.
   (b) $f(a+h) = f(a) + \langle \nabla f(a), h\rangle + \frac{1}{2}D^2 f(a)h + \|h\|^2 E(h)$.

5. Suppose $U \subseteq \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}^m$ is a smooth function. Moreover, assume that there exist $(x_0, y_0) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ such that $f(x_0, y_0) = 0$ and $D_{\mathbb{R}^m} f(x_0, y_0)$ is invertible. Let
$$M = \{(x, y) \in U : f(x, y) = 0\}.$$
Let $p \in M$. Prove that there exists an open set U in $\mathbb{R}^{n-m}$ and a smooth map $\psi : U \to \mathbb{R}^n$ such that $\psi(U)$ is an open subset of M containing p, ψ is one-one and has moreover, $D\psi(x)$ has rank $n - m$ for all $x \in U$.

Compute the orientation vector field along the parametrization (U, ψ) where $U = \{(\theta, \phi) \in \mathbb{R}^2 : 0 < \theta < \frac{\pi}{2}, 0 < \phi < \frac{\pi}{2}\}$ and $\psi : U \to \mathbb{R}^3$ is defined as

$$\psi(\theta, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi).$$

7. (a) Prove that if ω is a k-form on an open subset U of $\mathbb{R}^n$, k being odd, then $\omega \wedge \omega = 0$.

   (b) Suppose the coordinates in $\mathbb{R}^4$ are given by x_1, x_2, y_1, y_2. Consider the 2 form in $\mathbb{R}^4$ defined by
   $$\omega = d(x_1) \wedge d(y_1) + d(x_2) \wedge d(y_2).$$
   Then prove that $\omega \wedge \omega \neq 0$.

8. Let (U, ψ) be a parametrized 2-surface in $\mathbb{R}^3$ and let X_1, X_2 denote the coordinate vector fields along ψ. We define three functions E, F, G on U as
$$E = \langle X_1, X_1 \rangle, G = \langle X_2, X_2 \rangle, F = \langle X_1, X_2 \rangle,$$
i.e, for $p \in U$, $E(p) = \langle X_1(p), X_1(p) \rangle_{T_{\psi(p)}(\psi(U))}$, etc.
Then prove that
$$\mathrm{Vol}(\psi(U)) = \int_U \sqrt{EG(u_1, u_2) - F^2(u_1, u_2)}du_1 du_2.$$

9. Construct an orientation form on a regular n-level surface in $\mathbb{R}^{n+2}$ and compute the formula for the corresponding volume form.

10. Consider the trapezium with vertices $(a, 0), (b, 0), (e, f), (c, d)$. . Here, $b > a$, $e > a$, $f > 0$, $c < b$ and $d > 0$. Moreover, let $R := [0, 1] \times [0, 1]$. Consider the parametrization of the boundary of the trapezium given by
$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 : \partial R \to \mathbb{R}^2, \text{ where,}$$
$$\begin{aligned} \gamma_1(t, 0) &= (1-t)(a, 0) + t(b, 0) \\ \gamma_2(1, t) &= (1-t)(b, 0) + t(c, d) \\ \gamma_3(1-t, 1) &= (1-t)(c, d) + t(e, f) \\ \gamma_4(0, 1-t) &= (1-t)(e, f) + t(a, 0). \end{aligned}$$
Parametrize the interior of the trapezium by
$$\psi(x, y) = (1-x)\gamma_4(0, y) + x\gamma_2(1, y).$$
Using the Green's theorem for the rectangle, prove the Green's theorem for the trapezium.