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Indian Statistical Institute
B. STAT. II (2024-25)
2^{nd} Semester Examination 2025

Paper: **Differential Equations** Date: **09/05/2025**

Time : **3 Hrs.** Full Marks : **50**

Note: Symbols used have their usual meaning. *The figures in the margin indicate full marks.*
Total score will be calculated by the marks obtained in the TEN highest-scoring answered questions.

1. Determine the differential equation corresponding to the family of curves $y = c(x-c)^2$ and analyze its singular solutions. [5]

2. Find the general solution of the differential equation
$$y(y-2)y'^2 + (2x - xy - y)y' + x = 0$$
and obtain a particular solution valid on the whole real line and passing through the point $(-2/9, 0)$.
[5]

3. Solve the differential equation $(D-2)^2 y = x^2 e^{2x} \sin 2x$, where $D \equiv \dfrac{d}{dx}$. [5]

4. Let us consider a linear system controlled by the ordinary differential equation
$$ay''(t) + by'(t) + cy(t) = g(t).$$
Here a, b, c are real constants. Prove or disprove the fact:
"*The transfer function $H(s) = \dfrac{Y(s)}{G(s)}$ depends only on choice of a, b, c but not on the input function g*".
Here $Y = \mathcal{L}[y]$ and $G = \mathcal{L}[g]$ denote the Laplace transformation of $y(t)$ and $g(t)$ respectively. [5]

5. Consider the differential equations
$$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right] + Q_1(t)x = 0 \quad (1)$$
$$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right] + Q_2(t)x = 0, \quad (2)$$
where $P(t) > 0$ has a continuous derivative and Q_1 and Q_2 are continuous and $Q_2(t) > Q_1(t)$ on $a \le t \le b$. Let $a \le t_0 < t_1 \le b$ and ϕ_i be the solution of Equation (i) such that $\phi_i(t_0) = c_0, \phi_i'(t_0) = c_1$ for $i = 1, 2$, where $c_0 \ge 0$ and $c_1 > 0$ if $c_0 = 0$. Suppose $\phi_2(t) > 0$ for $t_0 < t < t_1$. Show that $\phi_1(t) > \phi_2(t)$ for $t_0 < t < t_1$. [5]

6. Let Φ be a solution of the non-homogeneous system

$$Y'(t) = AY(t) + G(t),$$

where A is an $n \times n$ constant matrix, $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}$ and $G(t)$ is $n \times 1$ matrix with a period $\omega > 0$ i.e., $G(t+\omega) = G(t)$. Prove that the system has a unique solution with period ω if and only if $Y'(t) = AY(t)$ has no solution with period ω except $Y = 0$. [5]

7. If $x = \infty$ is a regular singular point of the hypergeometric equation

$$x\left(1 - \frac{x}{b}\right) y'' + \left[(c - x) - \frac{(a+1)x}{b}\right] y' - ay = 0,$$

then show that $x = \infty$ is an irregular singular point of the differential equation

$$xy'' + (c - x)y' - ay = 0. \tag{3}$$

Find the general solution of the differential equation (3) near its regular singular point. [1 + 4]

8. Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = -Ae^{-\alpha x},$$

where $A \geq 0$ and $\alpha > 0$, and the given boundary conditions are

$$\begin{aligned} u(0, y) &= 0, \quad y > 0, \\ u(L, y) &= 0, \quad y > 0, \\ u(x, 0) &= \sin x, \ 0 < x < L. \end{aligned}$$

[5]

9. Find the general solution of differential equation

$$(12x + 29y)z\,dx - (11x + 12y)z\,dy - (2x^2 + 3xy - 2y^2)dz = 0$$

and determine the integral surface which passes through the curve $y = 0$, $z = x^6$. [5]

10. Consider the nonlinear autonomous system

$$\begin{aligned} \frac{dx}{dt} &= 4x + 4y - x(x^2 + y^2) \\ \frac{dy}{dt} &= -4x + 4y - y(x^2 + y^2). \end{aligned}$$

(a) Transform the system into polar coordinates. Then, using the Poincaré-Bendixon theorem, show that there exists a closed trajectory lying between the circles $r = 1$ and $r = 3$.

(b) Find the general non-constant solution $x = x(t)$ and $y = y(t)$ of the original system, and use this to find a periodic solution corresponding to the closed path whose existence was established in (a).

[2 + 3]

11. Classify all the bifurcations and plot the bifurcation diagram for $-\infty < r < \infty$ and indicate the stability of the various branches of fixed points for the system $\dot{x} = rx - \sin x$. [5]

12. Find and classify the fixed points of the following Lorenz system

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz, \end{aligned}$$

where $\sigma, r, b > 0$. Examine the stability of the fixed point at origin. [5]

13. Determine the curve of fixed length l that joins the points $(0, 0)$ and $(0, 2)$, lies above x-axis, and encloses the maximum area between itself and the x-axis. [5]