

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: (2024-2025)

M. Stat 2nd Year

Statistical Computing II

Date: ~~7/5/25~~ Marks: 100 Duration: 3 hours.

Attempt all questions

1. Let $f(x)$ and $g(x)$ be integrable real-valued functions of x . Let $\hat{f}$ and $\hat{g}$ be Fourier transforms of f and g . Then under further suitable assumptions, to be stated by you, prove that

(a) $\int f(x)g(x)dx = \frac{1}{2\pi} \int \hat{f}(y)\hat{g}(y)dy$.

- (b) Using the above result, prove that

$$\int \left| \frac{\sin(by)}{by} \right|^2 dy = \frac{\pi}{b}, \text{ for } b > 0.$$

- (c) Obtain the Fourier series expansion of $f(t) = |t|$ on $[-\pi, \pi)$.

[5 + 8 + 12 = 25]

2. Let $\mathbf{z} = (1, 0, -1, 0, 1, 0, -1, 0)$. Let the father and mother wavelets $\mathbf{u}$ and $\mathbf{v}$ be given by

$$\hat{\mathbf{u}} = (\sqrt{2}, \sqrt{2}, 0, 0, 0, 0, \sqrt{2}, \sqrt{2})$$

and

$$\hat{\mathbf{v}} = (0, 0, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, 0, 0),$$

respectively, where $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ denote Fourier transforms of $\mathbf{u}$ and $\mathbf{v}$, respectively. Then derive the second-stage averaging and detailing of $\mathbf{z}$ using the Folding Lemma, using $u_2(n) = u(n) + u(n + N/2)$ and $v_2(n) = v(n) + v(n + N/2)$; $n = 1, \dots, N/2$, where N , divisible by 2, denotes the relevant dimension.

[25]

3. Let $\psi \in L^2(\mathbb{R})$ where $L^2(\mathbb{R}) = \{f : \mathbb{R} \mapsto \mathbb{C} : \int_{\mathbb{R}} |f(x)|^2 < \infty\}$, where $\mathbb{R}$ and $\mathbb{C}$ are the real line and the complex space, respectively. For $t \in \mathbb{R} \setminus \{0\}$ and $g \in \mathbb{R}$, let

$$Wf(t, g) = \langle f, \psi_t^g \rangle = \frac{1}{\sqrt{|t|}} \int_{\mathbb{R}} f(x) \overline{\psi\left(\frac{x-g}{t}\right)} dx,$$

where for any $g \in L^2(\mathbb{R})$, $\overline{g(x)}$ denotes the complex conjugate of $g(x)$. Then prove that for any $f, g \in L^2(\mathbb{R})$,

$$(a) \quad |Wf(t, g)| \leq \|f\| \text{ for all } t, g \in \mathbb{R} \text{ with } t \neq 0.$$

$$(b) \quad \langle f, g \rangle = \frac{8\pi^3}{C_\psi} \int_{\mathbb{R} \setminus \{0\}} \int_{\mathbb{R}} Wf(t, g) \overline{Wg(t, g)} dy \frac{dt}{t^2}.$$

[5+20=25]

4. Consider support vector machine (SVM) for overlapping two-class distributions.

- Defining appropriate slack variables derive a criterion to be minimized, with appropriate constraints, for obtaining SVM.
- Formulate the constrained dual minimization problem.
- Is the dual minimization a convex problem? Justify.
- Derive the Karush-Kuhn-Tucker conditions.
- Analyse sparsity of the resultant SVM.

[5+5+5+5+5=25]