

Measure Theory

M. Math. 1st year
Back paper exam
3 hours 30 minutes
Full marks: 100

17 December 2025

Answer **all** questions. Each question carries 20 marks.

1. Suppose that $\Omega \neq \emptyset$, $\mathcal{F}$ is a field on Ω and $\mu : \mathcal{F} \rightarrow [0, \infty)$ is a finitely additive function, that is,

$$\mu(A_1 \cup \dots \cup A_n) = \sum_{i=1}^n \mu(A_i) \text{ for all disjoint } A_1, \dots, A_n \in \mathcal{F}.$$

Assume that if $A_n \in \mathcal{F}$ are such that $A_n \downarrow \emptyset$, then $\mu(A_n) \downarrow 0$. Suppose $A_1, A_2, \dots \in \mathcal{F}$ are disjoint such that

$$A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.$$

Define

$$B_n = A_n \cup A_{n+1} \cup \dots \text{ for all } n \geq 1.$$

- (a) [**8 marks**] Show that $B_n \downarrow \emptyset$ and hence argue that

$$\mu(B_n) \downarrow 0.$$

- (b) [**8 marks**] Hence or otherwise, prove that

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i).$$

- (c) [**4 marks**] Hence or otherwise, show that μ can be extended to a measure on $(\Omega, \sigma(\mathcal{F}))$.

2. Suppose $(\Omega, \mathcal{A}, \mu)$ is a measure space and $f : \Omega \rightarrow \mathbb{R}$ is an integrable function.

(a) [10 marks] Show that

$$\lim_{n \rightarrow \infty} \int_{\{|f| > n\}} |f| d\mu = 0.$$

(b) [10 marks] Hence or otherwise, show that given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_A |f| d\mu \leq \varepsilon$$

whenever $A \in \mathcal{A}$ is such that $\mu(A) \leq \delta$.

3. Suppose $(\Omega, \mathcal{A}, \mu)$ is a finite measure space and $f : \Omega \rightarrow [0, \infty)$ is an $\mathcal{A}$ -measurable function. Let $\mathcal{F} \subset \mathcal{A}$ be a σ -field. Assume that there exist a sequence α_n such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \dots$ and $\alpha_n \uparrow \infty$, which satisfy

$$[f \leq \alpha_n] \in \mathcal{F} \text{ for all } n \geq 0.$$

(a) [10 marks] For n fixed, define a measure ν_n on $(\Omega, \mathcal{F})$ by

$$\nu_n(A) = \int_{A \cap [\alpha_{n-1} < f \leq \alpha_n]} f d\mu, \quad A \in \mathcal{F}.$$

Show that $\nu_n(\Omega) < \infty$ and $\nu_n \ll \mu$ and hence argue that there exists $g_n \in L^1(\Omega, \mathcal{F}, \mu)$ with $g_n \geq 0$ such that

$$\int_A g_n d\mu = \nu_n(A) \text{ for all } A \in \mathcal{F}.$$

(b) [4 marks] Show that for all $n \geq 1$,

$$\int_A g_n d\mu = \int_{A \cap [\alpha_{n-1} < f \leq \alpha_n]} f d\mu, \quad A \in \mathcal{F}.$$

(c) [4 marks] Let

$$g = \sum_{n=1}^{\infty} g_n.$$

Show that

$$\int_A g d\mu = \int_A f d\mu \text{ for all } A \in \mathcal{F}.$$

(d) [2 marks] Prove that $g < \infty$ μ -a.e.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and $f : \Omega \rightarrow \overline{\mathbb{R}}$ be a measurable function.

- (a) **[3 marks]** If $\|f\|_\infty = 0$, show that $f = 0$ a.e. and hence argue that $\|f\|_p = 0$ for all $1 \leq p < \infty$.
- (b) **[5 marks]** Assume $\|f\|_\infty > 0$ and fix $0 < \alpha < \|f\|_\infty$. Define $A = \{|f| > \alpha\}$ and show that

$$\int |f|^p d\mu \geq \alpha^p \mu(A).$$

- (c) **[4 marks]** Hence or otherwise, prove that

$$\liminf_{p \rightarrow \infty} \|f\|_p \geq \|f\|_\infty.$$

- (d) **[6 marks]** Show that $\|f\|_p \leq \|f\|_\infty \mu(\Omega)^{1/p}$ for all $1 \leq p < \infty$ and hence argue that

$$\limsup_{p \rightarrow \infty} \|f\|_p \leq \|f\|_\infty.$$

- (e) **[2 marks]** Hence or otherwise, show that

$$\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty.$$

5. Suppose $(\Omega, \mathcal{A})$ is a measurable space on which μ and ν are σ -finite measures such that $\mu \ll \nu$. Let

$$f = \frac{d\mu}{d\nu},$$

that is, $f : \Omega \rightarrow [0, \infty)$ is a measurable function satisfying

$$\mu(A) = \int_A f d\nu \text{ for all } A \in \mathcal{A}.$$

Assume that $f > 0$.

- (a) **[8 marks]** Define

$$g = \frac{1}{f}.$$

Show that for all $A \in \mathcal{A}$,

$$\int_A g d\mu = \int_A g f d\nu.$$

- (b) **[6 marks]** Hence or otherwise, show that

$$\int_A g d\mu = \nu(A) \text{ for all } A \in \mathcal{A}.$$

- (c) **[6 marks]** Hence or otherwise, prove that $\nu \ll \mu$ and

$$g = \frac{d\nu}{d\mu}.$$