

Indian Statistical Institute
Semester 2 (2024-2025)
B. Stat 2nd Year
Semestral Examination
Introduction to Stochastic Processes

Monday, 5.5.2025, Time: 3 hours

Total Points: $5 \times (7 + 7) = 70$ pts.

Answers must be justified with clear and precise arguments. More than one uncrossed answers to the same question or part of a question will not be entertained and only the first uncrossed answer will be graded. If you use any theorem or result proved in class state it explicitly.

1. Consider a Markov chain $X_0, X_1, X_2, \dots$ on state space $S = \{0, 1, 2, \dots, d\}$ such that $P(i, j) > 0, \forall i, j \in S$. Two letter words are observed from the above chain in the manner of $(X_0X_1), (X_1X_2), (X_2X_3), \dots$. One can easily verify that these two letter words also form a Markov chain with $P((X_{n+1}X_n) = (k, l) | (X_nX_{n-1}) = (i, j)) = \delta_{jk}P(k, l)$.

(a) Using the assumption on $P(\cdot, \cdot)$ show that the Markov chain of two letter words is irreducible and aperiodic.

(b) Suppose π is the stationary distribution of the original chain X_n . Heuristically when one observes two letter words in the above manner the long term probability of observing (i, j) is the long term probability π_i multiplied by $P(i, j)$. Is $\pi_i P(i, j)$ a stationary distribution for the chain of two letter words observed as above?

2. (a) Consider a birth and death chain on $S = \{0, 1, 2, \dots\}$ such that $P(x, x+1) = p_x, P(x, x-1) = q_x = 1 - p_x, p_0 = 1, 0 < p_x < 1, \forall x > 0$. It is known that this chain is transient or recurrent according as $S_1 < \infty$ or $= \infty$ where

$$S_1 = 1 + \frac{q_1}{p_1} + \frac{q_1 q_2}{p_1 p_2} + \dots + \frac{q_1 \dots q_x}{p_1 \dots p_x} + \dots,$$

and positive recurrent iff $S_2 < \infty$ where

$$S_2 = 1 + \frac{p_0}{q_1} + \frac{p_0 p_1}{q_1 q_2} + \dots + \frac{p_0 p_1 \dots p_{x-1}}{q_1 q_2 \dots q_x} + \dots.$$

Using the n th term of an infinite series of positive terms (and using methods of infinite series only) show that $S_1 < \infty$ implies $S_2 = \infty$. Next using series and logical reasoning show that $S_2 < \infty$ implies $S_1 = \infty$. Can one see this last statement from the n th term of S_2 and infinite series methods only?

(b) Consider a Markov chain with state space $S = \{0, 1, 2, \dots\}$ such that $P(0, 1) = 1, P(1, 2) = P(1, 0) = 1/2, P(2n+1, 2n+2) = 1 - 1/(2n+1), P(2n+1, 2n) = 1/(2n+1), P(2n, 2n+2) = 1 - 1/(2n)^2, P(2n, 2n-1) = 1/(2n)^2, n = 1, 2, \dots$. Is this (clearly irreducible) MC positive recurrent, just recurrent or transient?

3. (a) If X is a nonnegative integer valued random variable prove that $EX = \sum_{k=0}^{\infty} P(X > k)$. (For nonnegative random variables with a general distribution function F the formula is $EX = \int_0^{\infty} P(X > u) du$, which can be assumed if not already proved. This is not used in this problem.)

(b) Consider a branching process starting from 1 particle whose progeny distribution (satisfying the usual conditions $f(0) > 0, f(0) + f(1) < 1$) has mean μ . If $\mu < 1$ then is $E_1 T_0 < \infty$?

4. (a) Suppose N_1 and N_2 are two independent Poisson processes on $(0, \infty)$ with parameters λ_1 and λ_2 respectively. Find the probability that the first arrival of N_2 is later than the second arrival of N_1 .

(b) Customers arrive and buy a certain brand of car battery from a store in the manner of a Poisson process with parameter λ and each car battery works for a length of time distributed with cdf F with mean μ . Find the distribution of the number of batteries still working at time t and find the limit of this distribution as $t \uparrow \infty$.

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5. (a) The Yule process is a simple birth process with $\lambda_n = \lambda n, n \geq 0$, that is $P_n(t) = P(X(t) = n)$ satisfies the forward equation $P'_n(t) = \lambda(n-1)P_{n-1}(t) - \lambda n P_n(t)$ ($P'_1(t) = -\lambda P_1(t)$) which using integrating factors becomes $e^{\lambda n t} P_n(t) - P_n(0) = \int_0^t \lambda(n-1) e^{\lambda n u} P_{n-1}(u) du$. If one starts from $X(0) = 1$, then with $P_1(0) = 1$, it is easily verified that $P_1(t) = e^{-\lambda t}$ and $P_2(t) = e^{-\lambda t}(1 - e^{-\lambda t})$. Prove **using integration by parts** the induction formula that $f_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$ satisfies ($f_n(0) = 1, n = 1, f_n(0) = 0, n > 1$)

$$e^{\lambda n t} f_n(t) - f_n(0) = \int_0^t \lambda(n-1) e^{\lambda n u} f_{n-1}(u) du.$$

- (b) Visitors arrive in front of 2 elevators in the ground floor following a Poisson process with parameter λ and they stand in queue. The time taken by each elevator to reach the ground floor follows independent $Exp(\mu)$ distributions and the capacity of each elevator is 4. Assume for convenience that as soon as an elevator arrives the process of boarding happens instantaneously and it leaves instantaneously (even with zero passengers). Modelling this queue length $X(t)$ as a continuous time Markov chain with state space $\{0, 1, 2, \dots\}$ find (the form of) the infinitesimal matrix for the forward equation $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{A}$ where $\mathbf{P}(t) = (P(X(t) = 0), P(X(t) = 1), \dots)$.