Indian Statistical Institute, Kolkata

M. Math I year

Backpaper Examination-Autumn 2025

Algebra I

Date: 15.12.2025

Full Marks: 100 *Time Allotted: 3 h*

Answer all questions.

(1) Let $n \in \mathbb{N}$. Show that there exists a surjective ring homomorphism

$$\mathbb{Z}[x] \longrightarrow \mathbb{Z}^n$$

if and only if $n \leq 2$. (5 + 5)

(2) Let R be a commutative ring with identity.

(a) Show that $R[[x]]$ is a PID.

(b) Show that every maximal ideal of $R[[X]]$ is of the form

$$\mathfrak{m} + (X),$$

where $\mathfrak{m}$ is a maximal ideal of R. (5 + 5)

(3) Let

$$A = \mathbb{C}[X,Y]/(X - Y^2), \quad B = \mathbb{C}[X,Y]/(XY - 1), \quad C = \mathbb{C}[X,Y]/(X^2 + Y^2 - 1).$$

Examine whether $A \cong B$, $B \cong C$, and $A \cong C$ as rings. (3 + 3 + 4)

(4) Let R be an integral domain. Let T be a torsion module over R, and let F be a free module over R. Show that $\mathrm{Hom}_R(T, F) = \{0\}$. Is this statement true if R is not an integral domain? (6 + 4)

(5) Let $p \in \mathbb{Z}$ be a prime, and let

$$M = \left\{ \frac{a}{p^n} \;\middle|\; a \in \mathbb{Z},\, n \in \mathbb{N} \right\}.$$

(a) Show that M is a $\mathbb{Z}$-module.

(b) Show that $M \otimes_{\mathbb{Z}} (M/\mathbb{Z}) = 0$. (5 + 5)

(6) Let $M = \mathbb{Z}/10 \oplus \mathbb{Z}/12$.

(a) Find all primes $p \in \mathbb{Z}$ such that $M_{(p)} \neq 0$.

(b) Describe $M_{(3)}$. (5 + 5)

(7) Find the invariant factors and the Jordan canonical form for the matrix

$$\begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -17 \\ 0 & 0 & 2 \end{pmatrix}$$

(5 + 5)

(8) Let G be a finite abelian group.

(a) Show that G is a torsion $\mathbb{Z}$-module.

(b) Suppose G is not cyclic. Show that G contains a subgroup isomorphic to $\mathbb{Z}/(p) \oplus \mathbb{Z}/(p)$ for some prime p.

(5 + 5)

(9) Let R be a commutative ring with unity and M a Noetherian R-module.

(a) Let $f : M \to M$ be a surjective R-module homomorphism. Show that f is an isomorphism.

(b) Show that $R/\mathrm{ann}_R(M)$ is a Noetherian ring. (5 + 5)

(10) State and prove the Hilbert Basis Theorem. (2 + 8)