

INDIAN STATISTICAL INSTITUTE KOLKATA
End-Semester Examination 2024-2025
MStat and MMath

Subject: Advanced Functional Analysis

Department: Stat-Math Unit

Full Marks: 55

Duration: 3 hrs.

Date of Exam: 28/04/2025

Special Instructions:

- This is not an open book examination.
- calculator is not allowed during the examination.
- Answer all the questions.

Question 1

Let X be a topological vector space. Then prove the following:

- (i) The convex hull of every open set is open.
- (ii) The sum of two closed sets may fail to be closed.
- (iii) Interior of a balanced set may not be balanced.

[2 × 3 = 6]

Question 2

Let $X = C(\mathbb{R}^n)$, the space of all continuous complex valued functions on $\mathbb{R}^n$. Let $K \subset \mathbb{R}^n$ be a compact set. The seminorm p_K associated to K is defined as

$$p_K(f) := \sup_{x \in K} |f(x)|.$$

Let $\mathcal{P}$ be the collection of seminorms p_K indexed by all compact subsets of $\mathbb{R}^n$. Show that $f_n \in X$, converges to f in the $\mathcal{P}$ -topology if and only if $f_n \rightarrow f$ uniformly over every compact subsets of $\mathbb{R}^n$. [8]

Question 3

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$ and $A = \{f \in C(\overline{\mathbb{D}}) : f \text{ is analytic on } \mathbb{D}\}$ and for $f \in A$ define $f^*(z) = \overline{f(\bar{z})}$.

Show that $f \rightarrow f^*$ is an involution on A and $\|f^*\| = \|f\|$ for all $f \in A$, but A is not a C^* -algebra. [8]

Question 4

Give an example of a C^* -algebra A and two positive elements a and b in A such that ab is not positive. If $ab = ba$ then show that $ab \in A_+$. [6]

Question 5

Let $\mathcal{H}$ be a separable Hilbert space and fix an orthonormal basis $\{e_n\}$ for $\mathcal{H}$.

- Show that 0 belongs to the weak closure of $\{\sqrt{n}e_n : n \geq 1\}$.
- Let n_i be a net of integers such that $\sqrt{n_i}e_{n_i} \rightarrow 0$ weakly and define $T_i f = \sqrt{n_i} \langle f, e_{n_i} \rangle e_{n_i}$ for all $f \in \mathcal{H}$. Show that $T_i \xrightarrow{\text{SOT}} 0$ but T_i^2 does not converge to 0 (in SOT).

[8]

Question 6

Show that a normal operator N is compact if and only if for every $\epsilon > 0$, $E(z : |z| > \epsilon)$ has finite rank, where E is the spectral measure. [6]

Question 7

Prove that the closed unit ball of L^1 (relative to Lebesgue measure on the unit interval) has no extreme points. [5]

Question 8

Suppose K is a compact convex set in $\mathbb{R}^n$. Prove that every $x \in K$ is a convex combination of at most $n + 1$ extreme points of K . [8]

—————End of the paper—————