

INDIAN STATISTICAL INSTITUTE, KOLKATA
BACKPAPER EXAMINATION: FIRST SEMESTER 2025 - '26
ML MATH I

Subject: Linear Algebra Date: December 19, 2025
Duration: 3 hrs Maximum score: 100

Attempt all the problems. Justify every step in order to get full credit of your answers. All arguments should be clearly mentioned on the answerscript. Points will be deducted for missing or incomplete arguments.

In this question paper, assume all vector spaces have finite dimension and characteristic of any field is NOT 2.

- (1) (a) Let X be an invertible complex $n \times n$ -matrix such that X^5 is a diagonalizable. Show that X is diagonalizable. Will the conclusion hold without the assumption of invertibility of X ? Justify. *Hints: Start working with the Jordan form of X .*
(b) Let A, B be invertible complex $n \times n$ -matrices such that $BA^2B = A^3$ and B has n -many distinct eigen values. Show that A is diagonalizable.
Hints: First show that A^5 and B commute with each other.

[(9+7)=16 marks]

- (2) Let V be a finite dimensional vector space over $\mathbb{F}$, $T \in L(V, V)$ and $p(x) \in \mathbb{F}[x]$ be monic irreducible such that V has a cyclic vector for T and $\chi_T = p^4$.
(a) Find a T -invariant subspace W of V such that $T|_W$ is simple,
(b) Justify that there does not exist any T -invariant subspace of V which is complementary to W . *Hints: Suppose the statement is false. Consider the minimal polynomial and arrive at a contradiction.*

[15 marks]

- (3) Let V be a finite dimensional complex inner product space and $T \in L(V, V)$.
(a) If T is normal, then show that dimensions of $\text{Range}(T)$ and $\text{Range}(T^*)$ are identical.
(b) Show that T is normal if and only if there exists unitary $U \in L(V, V)$ such that $T^* = UT$.

[16 marks]

- (4) Let $T \in L(V, V)$ be simple (that is, the only T -invariant subspaces of V are the two trivial ones).
(a) If $S \in L(V, V)$ such that $ST = TS$, then show that S is either zero or an isomorphism. *Hints: Establish injectivity of S .*
(b) Show that for all $v_1, v_2 \in V \setminus \{0\}$, there exists unique $S \in L(V, V)$ such that $Sv_1 = v_2$ and $ST = TS$.
(c) Find $\dim \{S \in L(V, V) : ST = TS\}$.

[16 marks]

(5) Let V, W be a finite dimensional complex inner product space.

(a) If $T \in L(V, W)$, then show that $\ker(T) = \ker(T^*T) = \text{Range}(T^*)^\perp$.

(b) Show that $\left(\text{span} \bigcup_{j=1}^k S_j\right)^\perp = \bigcap_{j=1}^k S_j^\perp$ for all $k \in \mathbb{N}$ and $S_1, \dots, S_k \subseteq V$.

(c) If $T_1, \dots, T_k \in L(V, V)$ are positive, then show that $\sum_{j=1}^k T_j$ is also positive and

$$\text{span} \bigcup_{m=1}^k \text{Range}(T_m) = \text{Range} \left(\sum_{j=1}^k T_j \right).$$

[15 marks]

(6) (a) Let A, B be two idempotents. Show that $A + B$ is idempotent if and only if $AB = 0 = BA$.

(b) Let P, Q be two orthogonal projections such that $\text{Range}(P) \subset \text{Range}(Q)$. Show that $Q - P$ is an orthogonal projection.

(c) Let $P_1, \dots, P_k$ be orthogonal projections such that $\sum_{j=1}^k P_j$ is also orthogonal projection. Show that $\text{Range}(P_j) \perp \text{Range}(P_m)$ for all $j \neq m$. *Hints: Mathematical induction on k and use (5)(a), (5)(c) and (6)(b).*

[16 marks]

(7) (a) Let $N_1, N_2 \in L(V, V)$ be nilpotent such that $N_1 N_2 = N_2 N_1$. Show that every element of $\{f(N_1, N_2) : f \in \mathbb{F}[x, y] \text{ such that } f(0, 0) = 0\}$ is nilpotent.

(b) If the minimum polynomial m_T of $T \in L(V, V)$ is irreducible, then how are $\deg(m_T)$ and $\dim(V)$ related? Justify.

(c) Give an example of a square complex matrix which is diagonalizable but not normal. Justify.

[15 marks]