Indian Statistical Institute
Final - Discrete Mathematics
BStat 2nd year, 3rd Semester

Time: 3 hours

25th April 2025

There are 11 problems with a total of 120 marks. You can attempt all the problems. The maximum score one can score is 100.

1. Let G be a graph with n vertices and m edges. Label each edge with a distinct number from $1, 2, \ldots, m$. Show that there exists an increasing path of length at least $\frac{2m}{n}$ (By an increasing path, we mean a path which follows edges with strictly increasing labels).

[10]

2. (a) If $T(n) = T(\lfloor n/2 \rfloor) + (\lfloor \log n \rfloor) + 9$ and $T(1) = 1$ then compute $T(n)$.
   (b) Is $T(n)$
      i. $\Omega(f(n))$ where
      $$f(n) = \begin{cases} n^{1/1000} & \text{when } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$
      ii. $\Theta(g(n) + \log n \log\log n)$, where
      $$g(n) = \begin{cases} \log n & \text{when } n \text{ is a power of 2} \\ 20 \log\log n & \text{otherwise} \end{cases}$$
      iii. $o(15 \log^2 n)$,
      iv. $O((\log\log n)!)$

[$7 + (2 \times 4) = 15$]

3. Assume a point p belongs to the convex hull of a finite set $Q \subset \mathbb{R}^d$. Show that there are d points $v_1, \ldots, v_d$ (some of them might coincide) of Q and d non-negative real numbers $\lambda_1, \ldots, \lambda_d$ such that
$$p = \sum_{i=1}^{d} \lambda_i v_i \quad \text{and} \quad \sum_{i}^{d} \lambda_i \leq 1.$$

[10]

4. Using any method, prove the following identity for all $n \in \mathbb{N} \setminus \{0\}$
$$n^n = \sum_{k=1}^{n} \binom{n}{k} k! k n^{n-k-1}$$

[10]

5. (a) You are given two trees, T_1 and T_2, each with 10 vertices and maximum degree 3. Both have exactly three leaves and exactly one vertex of degree 3. Prove or disprove: must T_1 and T_2 be isomorphic?

   (b) How many non-isomorphic trees exist with 6 vertices?

[5+5 = 10]

6. Let G be a simple undirected graph with $n \geq 3$ vertices and minimum degree at least $n/2$.

   (a) Prove that G is connected.

   (b) Prove that G contains a Hamiltonian cycle.

   (c) If G is also bipartite, prove that G has a perfect matching.

[4+5+6=15]

7. The *line graph* $L(G)$ of a graph G is the graph whose vertices correspond to the edges of G, with two vertices in $L(G)$ adjacent if and only if their corresponding edges in G share a common endpoint.

   (a) Draw a graph G such that $L(G)$ is isomorphic to K_3.

   (b) Prove that the line graph of a tree has no triangles.

[5+5 = 10]

8. If G is a tournament graph and x is a vertex with the maximum out-degree Show that for any other vertex in G, say y, there is a directed path from x to y of length at most 2.

[10]

9. Is it true that a strongly connected directed graph with equal in-degree and out-degree at every vertex must have an Eulerian path? Justify with a concise argument.

[10]

10. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two planar graphs.

    (a) Suppose $|V_1 \cap V_2| = 2$. Prove or disprove: the graph $H = (V_1 \cup V_2, E_1 \cup E_2)$ is planar.

    (b) Assume $V_1 \cap V_2 = \{v_1, v_2\}$, where $(v_1, v_2) \in E_1$ but $(v_1, v_2) \notin E_2$. Prove or disprove: the graph $H = (V_1 \cup V_2, E_1 \cup E_2)$ is planar.

    (c) Suppose $(v_1, v_2) \in E_1$ and $(v_1, v_2) \in E_2$. Prove or disprove: the graph $H = (V_1 \cup V_2, E_1 \cup E_2)$ is planar.

[4+4+2 = 10]

11. Let P be a set of n points in the plane. Show that there are at most $O(n^{7/3})$ triangles of unit area whose vertices are from P. (Hint: Use Szemerédi-Trotter theorem.)

[10]