INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination (Supplementary): 2024-25

B. Stat. I Year

Vectors and Matrices II

Date: 01/04/2025 Maximum Marks: 40 Duration: 2 Hours

Note: Give proper justification to all your answers.
State clearly all the results you are using.

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(1) Let M_2 be the set of all 2×2 matrices over $\mathbb{R}$ and $D : M_2 \to \mathbb{R}$ be a function such that $D(AB) = D(A)D(B)$ for all $A, B \in M_2$. Suppose [10]

$$D\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \neq D\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Prove the following.

(a) $D(0) = 0$,

(b) $D(A) = 0$ if $A^2 = 0$,

(c) $D(B) = -D(A)$ if B is obtained by interchanging two rows of A.

(d) $D(E) = 0$, where $E = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$, $a, b \in \mathbb{R}$.

[*Hint.* $E^2 = \left[\begin{smallmatrix} 0 & 0 \\ ab & b^2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 0 \\ a & b \end{smallmatrix}\right]$ and $D\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right]\right) = -D\left(\left[\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right]\right)$.]

(2) Let A be an $m \times n$ matrix and P, Q be orthogonal matrices of orders m and n respectively. Let $B = PAQ$. Show that [5]

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}^2,$$

where a_{ij} and b_{ij} denote the (i,j)th entry of A and B, respectively, $i, j = 1, 2, \ldots, n$.

(3) Consider $\mathbb{R}^4$ with the standard inner product. Let W be the subspace of $\mathbb{R}^4$ consisting of all vectors which are orthogonal to both $u = (-1, 2, 0, 1)$ and $v = (2, -3, 1, 2)$. Find a basis for W. [5]

(4) Let $x_1 = (3, 0, 3)$, $x_2 = (2, 0, 3)$, and $x_3 = (1, -1, -2)$. Apply the Gram Schmidt orthogonalization process to $\{x_1, x_2, x_3\}$ to obtain an orthonormal basis of $\mathbb{R}^3$ with the standard inner product. [5]

(5) Consider the subspaces S and T of $\mathbb{R}^5$, where

$$S = \{(x_1, x_2, x_3, x_4, x_5) : x_1 = x_3 = x_4 \text{ and } x_2 + x_5 = 0\}$$

and

$$T = \{(x_1, x_2, x_3, x_4, x_5) : x_1 = x_4 = x_5 \text{ and } x_3 = 0\}.$$

Equip $\mathbb{R}^5$ with the standard inner product. Verify that $(S + T)^\perp = S^\perp \cap T^\perp$. [8]

(6) Let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

(a) Find all the eigenvalues of A.

(b) Find a basis for the eigenspace corresponding to each eigenvalue of A. [9]
