

INDIAN STATISTICAL INSTITUTE

MID-SEMESTRAL EXAMINATION B.STAT - III YEAR

DESIGN AND ANALYSIS OF ALGORITHMS

Date: 12.09.2025 Maximum marks: 60 Duration: 2 hours

The paper contains 80 marks. Answer as much as you can, the maximum you can score is 60.

Note: $\mathbb{N}$ is the set of all natural numbers, and $\mathbb{R}$ is the set of all real numbers.

1. (a) Let f and g be monotone increasing functions from $\mathbb{N}$ to $\mathbb{R}^+$ such that $f = O(g)$ on the set $S = \{2^k : k \in \mathbb{N}\}$ and further, $g(2^{k+1}) \leq 10 \cdot g(2^k)$ for all $k \in \mathbb{N}$. Show that $f = O(g)$ over $\mathbb{N}$.
(b) Is $2^{n^2+n^{-1/2}} = O(2^{n^2})$? Justify your answer.

(5 + 5 = 10)

2. Consider the following algorithm on an input array $X[1, \dots, n]$.

```
1.  $j \leftarrow 1; m \leftarrow X[1]$ 
2. for  $k \leftarrow 2$  to  $n$  do
3.   if  $X[k] < m$  then
4.      $j \leftarrow k; m \leftarrow X[k]$ 
5.   end if
6. end for
```

Let N be the number of times Step 4 is executed.

- (a) What does the above algorithm do? Justify your answer.
- (b) Suppose the entries of the array X are from the set $\{1, \dots, r\}$. What are the minimum and the maximum values of N ?
- (c) Suppose the entries of the array X are independently and uniformly chosen bits. What is the expected value of N ?

(3 + 3 + 4 = 10)

3. (a) Let $I(T_n)$ and $E(T_n)$ be the internal and external path lengths of an extended binary tree T_n with n internal nodes. Show that $E(T_n) = I(T_n) + 2n$ for all $n \geq 0$.
(b) Show that if an extended binary tree with n internal nodes has all its external nodes at most at levels ℓ and $\ell + 1$, then $2^\ell \leq n + 1 < 2^{\ell+1}$.

(5 + 5 = 10)

4. (a) Construct a Huffman tree using the following numbers: 2, 6, 10, 13, 21, 27, 32, 41.
 (b) What data structures would you use to implement the algorithm for constructing a Huffman tree? Justify your answer.

(5 + 5 = 10)

5. (a) Write pseudocode for a modification of the quicksort algorithm to find the median of n numbers.
 (b) Derive the time complexity of your algorithm.

(5 + 5 = 10)

6. (a) Describe an algorithm to build a min heap on an array of n numbers.
 (b) Derive the time complexity of your algorithm.

(5 + 5 = 10)

7. (a) Provide a top level description (pseudocode not required) of Batchier's sorting algorithm.
 (b) Let $\text{CE}(a, b) = (\min(a, b), \max(a, b))$. Suppose the following sequence of CE operations is executed on input (x_1, x_2, x_3, x_4) .

$(x_2, x_3) \leftarrow \text{CE}(x_2, x_3)$
 $(x_1, x_2) \leftarrow \text{CE}(x_1, x_2)$
 $(x_3, x_4) \leftarrow \text{CE}(x_3, x_4)$
 $(x_2, x_3) \leftarrow \text{CE}(x_2, x_3)$
 $(x_1, x_2) \leftarrow \text{CE}(x_1, x_2)$
 $(x_3, x_4) \leftarrow \text{CE}(x_3, x_4)$.

Does this sequence of operations sort the input (x_1, x_2, x_3, x_4) ? Justify your answer.

(6 + 4 = 10)

8. (a) Describe the weighting and collapsing rule of the union-find algorithm.
 (b) What is the minimum number of union operations required to obtain a tree of height 4 using the above rules?

(6 + 4 = 10)